

The ρ -Capacity of a Graph

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Abstract

Motivated by the problem of zero-error broadcasting, we introduce a new notion of graph capacity, termed ρ -capacity, that generalizes the Shannon capacity of a graph. We derive upper and lower bounds on the ρ -capacity of arbitrary graphs, and provide a Lovász-type upper bound for regular graphs. We study the behavior of the ρ -capacity under two graph operations: the strong product and the disjoint union. Finally, we investigate the connection between the structure of a graph and its ρ -capacity.

I. INTRODUCTION

The zero-error capacity of a discrete memoryless noisy channel was first investigated by Shannon [1]. In this setup, a transmitter would like to communicate a message to a receiver through the channel, and the receiver must decode the message without error. The zero-error capacity is the supremum of all achievable communication rates under this constraint, in the limit of multiple channel uses. This problem can be equivalently cast in terms of the *confusion graph* G associated with the channel. The vertices of the confusion graph are the input symbols, and two vertices are adjacent if the corresponding inputs can result in the same output. Letting G^n denote the n th-fold strong product of G , which is the confusion graph for n uses of the channel, the zero-error capacity is obtained as the exponential growth rate of $\alpha(G^n)$, the size of a maximum independent set of G^n . The zero-error capacity is thus often referred to as the *Shannon capacity* of a graph, and is denoted by $C(G)$. Despite the apparent simplicity of the problem, a general characterization of $C(G)$ remains elusive. Lower and upper bounds were obtained by Shannon [1], Lovász [2] and Haemers [3].

Our work is motivated by the more general problem of characterizing the zero-error capacity of the discrete memoryless broadcast channel with two receivers. This problem can be cast in terms of the confusion graphs (G_1, G_2) corresponding to each of the receivers, and the associated capacity region is denoted by $C(G_1, G_2)$. This setup was considered by Weinstein [4], who found the region $C(G_1, G_2)$ in a few special cases:

- Both graphs are disjoint union of cliques. Note that the capacity region in this case can be deduced from more general results by Pinsker [5], Marton [6] and Willems [7].
- G_1 is the complete graph minus a clique, and G_2 is either empty or the complement graph of G_1 . The capacity region in this case is obtained by time sharing between the optimal point-to-point zero-error codes for G_1 and G_2 .

In this paper, we focus on the case where G_1 is the empty graph (i.e., the first receiver observes the input noiselessly), but where the graph G_2 can be arbitrary. This naturally gives rise to the notion of the ρ -capacity of a graph. Specifically, the ρ -capacity of the graph G_2 , written as $C_\rho(G_2)$, is the maximal rate that can be conveyed with zero-error to the second receiver, while communicating with the first (noiseless) receiver at a rate of at least ρ . In terms of the graph, the ρ -capacity is the exponential growth rate of the maximal number of pairwise non-adjacent subsets of size $2^{\rho n}$ in G^n . This notion of capacity generalizes the Shannon capacity of a graph, which is obtained as $C(G) = C_0(G)$.

Our paper is dedicated to the study of the ρ -capacity. In Section II, we formally define the ρ -capacity and explore its relation to the zero-error broadcasting problem. In Section III, we provide several upper and lower bounds on the ρ -capacity of arbitrary graphs, as well as an upper bound on the ρ -capacity of regular graphs that generalizes Lovász's construction [2], [8], [9]. In Section IV, we study the behavior of the ρ -capacity under two operations on graphs: the strong graph product and the disjoint graph union. Some relations between the ρ -capacity curve of a graph and its structure are investigated in Section V. We conclude by briefly discussing a few open problems in Section VI.

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II. PRELIMINARIES

A. Notations and Background

Let $G = (V, E)$ be a graph with vertex set V and edge set E . Two vertices v_1, v_2 are called adjacent if there is an edge between v_1 and v_2 , written as $v_1 \sim v_2$. An *independent set* in G is a subset of pairwise non-adjacent vertices. A *maximum independent set* is an independent set with the largest possible number of vertices. This number is called the *independence number* of G , and denoted by $\alpha(G)$. We write K_m for the complete graph over m vertices.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The *strong product* $G \boxtimes H$ of the graphs G and H is a graph such that

- 1) the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and
- 2) any two distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if $u \sim v$ and $u' = v'$, or $u = v$ and $u' \sim v'$, or $u \sim v$ and $u' \sim v'$.

For graphs G and H , we let $G + H$ denote their disjoint union. For a positive integer n , we interpret nG as the disjoint union of n copies of G . Two graphs G and H are called *isomorphic*, written as $G \cong H$, if there exists a bijection φ from $V(G)$ onto $V(H)$ such that any two vertices u and v in G are adjacent if and only if $\varphi(u)$ and $\varphi(v)$ in H are adjacent. Note that the strong product is commutative and associative in the sense that

$$\begin{aligned} G_1 \boxtimes G_2 &\cong G_2 \boxtimes G_1, \\ (G_1 \boxtimes G_2) \boxtimes G_3 &\cong G_1 \boxtimes (G_2 \boxtimes G_3). \end{aligned}$$

It is also immediate that the strong product is distributive for the disjoint union:

$$G_1 \boxtimes (G_2 + G_3) = G_1 \boxtimes G_2 + G_1 \boxtimes G_3.$$

(See [10, Section 5.2] for more properties of the strong product.) The graph $G^{\boxtimes n}$ is defined inductively by $G^{\boxtimes n} = G^{\boxtimes n-1} \boxtimes G$. For simplicity we will write G^n instead of $G^{\boxtimes n}$.

The *Shannon capacity* of a graph G is defined as the exponential growth rate of the independence number of G^n , i.e.,

$$C(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha(G^n), \quad (1)$$

where the limit exists by the superadditivity of $\log \alpha(G^n)$. This quantity also arises as the zero-error capacity in the context of channel coding [1], as we now briefly delineate.

A (point-to-point) *discrete memoryless channel* consists of a finite input alphabet \mathcal{X} , a finite output alphabet \mathcal{Y} , and a conditional probability mass function $p(y|x)$, such that $p(y^n|x^n) = \prod_{i=1}^n p(y_i|x_i)$ when the channel is used n times. A transmitter would like to convey a message¹ $w \in [2^{nR}]$ to a receiver over this channel, where the transmitter can set the input sequence x^n to the channel, and the receiver observes the output sequence y^n . To that end, the transmitter and receiver use an (n, R) *code*, which consists of an encoder $\psi : [2^{nR}] \rightarrow \mathcal{X}^n$ and a decoder $g : \mathcal{Y}^n \rightarrow [2^{nR}]$. Such a code is said to be *zero-error* if w can always be *uniquely determined* from y^n , i.e., $w = g(y^n)$ for any w and any correspondingly feasible output sequence y^n . We say that the communication rate R is *achievable* if an (n, R) zero-error code exists² for some n . The *zero-error capacity* of the channel is defined to be the supremum of all achievable rates.

The channel $p(y|x)$ can be associated with a *confusion graph* G , whose vertex set is the input alphabet \mathcal{X} , and whose edge set consists of all input pairs (x, x') that can lead to the same output, i.e., for which both $p(y|x) > 0$ and $p(y|x') > 0$ for some $y \in \mathcal{Y}$. It is easy to verify that G^n is the confusion graph associated with the product channel $p(y^n|x^n)$. It is well known and easy to check that the zero-error capacity of the channel is equal to $C(G)$, the Shannon capacity of its confusion graph G .

¹Throughout the paper we ignore integer issues whenever they are not important.

²Note that since a concatenation of two zero-error codes is a zero-error code, R is achievable for arbitrarily large n .

B. ρ -Capacity

In this subsection we introduce the ρ -capacity of a graph, which is a generalization of the Shannon capacity. We begin by generalizing the notion of an independent set of a graph. Let $G = (V, E)$ be a graph with vertex set V and edge set E . Two disjoint subsets of vertices V_1, V_2 are called *adjacent* if there exist vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1 \sim v_2$. Let $\mathcal{F} = \{V_i : 1 \leq i \leq l\}$ be a family of disjoint subsets $V_i \subset V$. If they are pairwise non-adjacent, then we call \mathcal{F} an *independent family* of G . Moreover, if each V_i is of size not less than k , then we say \mathcal{F} is a *k -independent family* of G . We write $|\mathcal{F}|$ for the number of subsets in \mathcal{F} . A *maximum k -independent family* is a k -independent family with the largest possible number of subsets. This number is called the *k -independence number* of G , and denoted by $\alpha_k(G)$. In particular, we have $\alpha_1(G) = \alpha(G)$, where $\alpha(G)$ is the independence number of G .

Example 1. Let C_5 be the pentagon graph, whose vertex set is $\{1, 2, 3, 4, 5\}$ and edge set is $\{12, 23, 34, 45, 51\}$. Then $\mathcal{F} = \{\{1, 2\}, \{4\}\}$ is an independent family of C_5 , and it is easy to verify that $\alpha_2(C_5) = 1$.

Here is a simple property of the k -independence number.

Lemma 1. *Let G and H be two graphs, and let k_1 and k_2 be two positive integers. Then*

$$\alpha_{k_1 k_2}(G \boxtimes H) \geq \alpha_{k_1}(G) \cdot \alpha_{k_2}(H).$$

Proof: Suppose that $\{V_i : 1 \leq i \leq l\}$ is a k_1 -independent family of G , and $\{U_j : 1 \leq j \leq l'\}$ is a k_2 -independent family of H . Then their Cartesian product $\{V_i \times U_j : 1 \leq i \leq l, 1 \leq j \leq l'\}$ is a $k_1 k_2$ -independent family of $G \boxtimes H$, and the result follows. \blacksquare

We now define the ρ -capacity of a graph G to be the exponential growth rate of the $2^{\rho n}$ -independence number of G^n , which generalizes the expression (1) for the Shannon capacity.

Definition 1. *Let G be a graph with m vertices. Then for any $0 \leq \rho \leq \log m$, the ρ -capacity of G is defined to be*

$$C_\rho(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_{2^{\rho n}}(G^n). \quad (2)$$

In particular, the ρ -capacity for $\rho = 0$ is equal to the Shannon capacity of the graph, i.e., $C_0(G) = C(G)$.

Note that the existence of the limit (2) follows from the superadditivity of $\log \alpha_{2^{\rho n}}(G^n)$, namely

$$\log \alpha_{2^{\rho(n+n')}}(G^{n+n'}) \geq \log \alpha_{2^{\rho n}}(G^n) + \log \alpha_{2^{\rho n'}}(G^{n'}),$$

which is guaranteed by Lemma 1. In particular, it also holds that $C_\rho(G) = \sup\{\frac{1}{n} \log \alpha_{2^{\rho n}}(G^n) : n = 1, 2, \dots\}$.

C. A Zero-Error Broadcasting Formulation

In this subsection, we show how the ρ -capacity arises naturally in the context of zero-error broadcasting. A (two-user) *discrete memoryless broadcast channel* consists of a finite input alphabet \mathcal{X} , two finite output alphabets \mathcal{Y}_1 and \mathcal{Y}_2 , and a conditional probability mass function $p(y_1, y_2|x)$, such that $p(y_1^n, y_2^n|x^n) = \prod_{i=1}^n p(y_{1i}, y_{2i}|x_i)$ when the channel is used n times. A transmitter would like to convey two messages $w_1 \in [2^{nR_1}]$ and $w_2 \in [2^{nR_2}]$ to two receivers over this channel, where the transmitter can set the input sequence x^n to the channel, and the receivers observe their respective output sequences y_1^n and y_2^n . To that end, the transmitter and receivers use an (n, R_1, R_2) code, which consists of an encoder $\psi : [2^{nR_1}] \times [2^{nR_2}] \rightarrow \mathcal{X}^n$, and two decoders $g_1 : \mathcal{Y}_1^n \rightarrow [2^{nR_1}]$ and $g_2 : \mathcal{Y}_2^n \rightarrow [2^{nR_2}]$. Such a code is said to be *zero-error* if w_1 and w_2 can always be *uniquely determined* from y_1^n and y_2^n , i.e., $w_1 = g_1(y_1^n)$ and $w_2 = g_2(y_2^n)$ for any w_1, w_2 and any correspondingly feasible pair of output sequences. We say that the communication rates (R_1, R_2) are *achievable* if an (n, R_1, R_2) zero-error code exists for some n . The *zero-error capacity region* of the broadcast channel is the closure of the set of all achievable rates.

Similarly to the case of the broadcast capacity under a vanishing-error criterion, it is easy to observe the following result.

Proposition 1. *The zero-error capacity region of a broadcast channel depends only on the conditional marginal distributions $p(y_1|x)$ and $p(y_2|x)$.*

Let $G_1 = (\mathcal{X}, E_1)$ and $G_2 = (\mathcal{X}, E_2)$ be the confusion graphs associated with the channels $p(y_1|x)$ and $p(y_2|x)$ respectively. Then Proposition 1 implies a simple corollary.

Corollary 1. *The zero-error capacity region of a broadcast channel depends only on the confusion graphs G_1 and G_2 .*

Following this, we write $C(G_1, G_2)$ for the zero-error capacity region of a broadcast channel with confusion graphs G_1 and G_2 . We now show that the ρ -capacity of G is the maximal rate that can be conveyed under zero-error to a noisy receiver with confusion graph G , while at the same time communicating with a noiseless receiver (i.e., having an empty confusion graph) at a rate of at least ρ .

Proposition 2. *Let G be a graph over m vertices. Then for any $0 \leq \rho \leq \log m$, we have*

$$C_\rho(G) = \sup \{R : (\rho, R) \in C(\overline{K}_m, G)\}$$

where \overline{K}_m is the empty graph over m vertices.

Proof: Let \mathcal{F} be a $2^{\rho n}$ -independent family of G^n , and set $R = \frac{1}{n} \log |\mathcal{F}|$. Then \mathcal{F} induces an (n, ρ, R) zero-error code for the broadcast setup associated with the definition of ρ -capacity, i.e., where the first receiver is noiseless and the second receiver has confusion graph G . The (n, ρ, R) zero-error code is constructed using *superposition coding*: the transmitter chooses a subset of \mathcal{F} for the second receiver, and chooses a vertex inside that subset for the first receiver, which is then transmitted. Clearly, the second receiver can always distinguish between the subsets of \mathcal{F} (hence decode its message with zero-error), whereas the first receiver can decode both messages with zero-error. Therefore $(\rho, C_\rho(G)) \in C(\overline{K}_m, G)$, and hence $C_\rho(G) \leq \sup \{R : (\rho, R) \in C(\overline{K}_m, G)\}$.

Conversely, suppose that the rate pair (ρ, R) is achievable, i.e., there exists an (n, ρ, R) zero-error code for some n . Consider the subsets of codewords obtained by fixing the second receiver's message and going over all the messages of the first receiver. All these subsets are of size $2^{\rho n}$, and since the second receiver must decode with zero-error regardless of the first receiver's message, any pair of these subsets must be non-adjacent in G^n . This naturally induces a $2^{\rho n}$ -independent family of G^n whose number of subsets is equal to 2^{nR} . Therefore $R \leq C_\rho(G)$, and hence $C_\rho(G) \geq \sup \{R : (\rho, R) \in C(\overline{K}_m, G)\}$. This concludes our proof. ■

The following proposition shows how the ρ -capacity can be used to provide a partial characterization of the zero-error broadcast capacity region.

Proposition 3. *Let $G_1 = (\mathcal{X}, E_1)$ and $G_2 = (\mathcal{X}, E_2)$ be two graphs. Let \mathfrak{C} be the convex hull of the closure of all rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_1 &\leq \rho/s, \\ R_2 &\leq C_\rho(H)/s, \end{aligned}$$

where H is the induced subgraph of G_2^s associated with some independent set A of G_1^s for some positive integer s , and $0 \leq \rho \leq \log |A|$. Then $\mathfrak{C} \subseteq C(G_1, G_2)$. Moreover, if $E_1 \subseteq E_2$ then $\mathfrak{C} = C(G_1, G_2)$.

Proof: Fix a $\rho \in [0, \log |A|]$. Then there exists a sequence of $2^{\rho n}$ -independent families \mathcal{F}_n of H^n such that $\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{F}_n| = C_\rho(H)$. Using superposition coding, we can see that \mathcal{F}_n is an $(sn, \rho, \frac{1}{n} \log |\mathcal{F}_n|)$ zero-error code. Hence $\mathfrak{C} \subseteq C(G_1, G_2)$. The second statement can be proved similarly as in Proposition 2. ■

D. Simple Properties of the ρ -Capacity

Proposition 4. *Let G be a graph with m vertices. The following properties of its ρ -capacity are easily observed:*

- 1) $C_0(G) = C(G)$, i.e., the ρ -capacity for $\rho = 0$ is equal to the Shannon capacity of the graph.
- 2) $C_{\log m}(G) = 0$.
- 3) $C_\rho(G)$ is monotonically non-increasing in ρ on $[0, \log m]$ (by definition).

4) $C_\rho(G)$ is concave in ρ on $[0, \log m]$ (by time sharing).

We now define three quantities related to the ρ -capacity, which will be of interest in the sequel. Let G be a graph over m vertices. We write $\rho^*(G)$ for the maximal $\rho \in [0, \log m]$ such that $C_\rho(G) = C(G)$, and $\rho_*(G)$ for the minimal $\rho \in [0, \log m]$ such that $C_\rho(G) = \log m - \rho$. We refer to $\rho^*(G)$ as the *free-lunch point* of G , and to $\rho_*(G)$ as the *packing point* of G . The *concave conjugate* of $C_\rho(G)$ is defined as

$$C_*(G, \gamma) \triangleq \inf_{\rho \in [0, \log m]} \gamma \rho - C_\rho(G) \quad \text{for } \gamma \in [-1, 0].$$

Here are two simple bounds on the ρ -capacity in terms of the Shannon capacity $C(G)$. The lower bound follows by time sharing (concavity), and the upper bound follows from the definition of an independent family.

Proposition 5. *Let G be a graph with m vertices. Then, for $0 \leq \rho \leq \log m$, we have*

$$\frac{C(G)}{\log m} (\log m - \rho) \leq C_\rho(G) \leq \min\{C(G), \log m - \rho\}. \quad (3)$$

Example 2. Let G be a disjoint union of two cliques, each of size $\frac{m}{2}$. It is easy to see that $C(G) = 1$, and that $\alpha_{\frac{m}{2}}(G) = 2$. Hence, $C_\rho(G) \geq 1$ for any $\rho \in [0, \log \frac{m}{2}]$, and by concavity also $C_\rho(G) \geq \log m - \rho$ for any $\rho \in [\log \frac{m}{2}, \log m]$. Hence the upper bound from Proposition 5 is tight in this case. In particular, the free-lunch point and the packing point coincide, $\rho^*(G) = \rho_*(G) = \log \frac{m}{2}$. The concave conjugate is given by $C_*(G, \gamma) = \gamma \rho_*(G) - C_0(G) = \gamma \log \frac{m}{2} - 1$.

Example 3. [4, Theorem 7] Let G be the complete graph on m vertices minus a clique on d vertices. Then

$$C_\rho(G) = \log d - \frac{\log d}{\log m} \rho.$$

This meets the lower bound of Proposition 5. In particular, the free-lunch point $\rho^*(G) = 0$ and the packing point $\rho_*(G) = \log m$. The concave conjugate is given by $C_*(G, \gamma) = \min\{-\log d, \gamma \log m\}$.

III. BOUNDS ON THE ρ -CAPACITY

In this section, we give three types of bounds on the ρ -capacity of a graph. The first bound is trivially derived from the capacity region of the degraded broadcast channel under the vanishing-error criterion. The second is based on the distribution of independent families and clique covers, via an explicit expression for the ρ -capacity of a disjoint union of cliques. The third generalizes Lovász's ϑ -function upper bound for the Shannon capacity.

A. An Information-Theoretic Upper Bound

The random variables X, Y, Z are said to form a *Markov chain* in that order, denoted by $X - Y - Z$, if their joint probability mass function can be written as $p(x, y, z) = p(x)p(y|x)p(z|y)$.

Theorem 1. *The ρ -capacity of a graph G satisfies*

$$C_\rho(G) \leq \min_{p(y|x)} \max_{\substack{U-X-Y \\ H(X|U) \geq \rho}} I(U; Y)$$

where the min is taken over all possible point-to-point channels $p(y|x)$ associated with a confusion graph G , and the random variable U has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}|\}$.

Proof: Consider a broadcast channel where the first receiver sees a noiseless channel, i.e., observes the input x , and the second receiver sees the input x through a noisy channel $p(y|x)$. The capacity region for this broadcast channel under the vanishing-error criterion is the convex hull of the closure of all (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq H(X|U), \\ R_2 &\leq I(U; Y) \end{aligned} \quad (4)$$

for some Markov chain $U - X - Y$, where the auxiliary random variable U has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}|\}$ (see [11, Theorem 15.6.2]). In particular, if $p(y|x)$ has confusion graph G , then the above region contains the zero-error capacity region $C(\overline{K}_m, G)$ where $m = |\mathcal{X}|$. The result now follows from Proposition 2. ■

B. Upper and Lower Bounds Based on Disjoint Union of Cliques

Recall that the Rényi entropy of order β , where $\beta \geq 0$ and $\beta \neq 1$, is defined as

$$H_\beta(P) = \frac{1}{1-\beta} \log \sum_{i=1}^s p_i^\beta,$$

where $P = \{p_1, \dots, p_s\}$ is a probability distribution. The limiting value of H_β as $\beta \rightarrow 1$ is the Shannon entropy $H_1(P) = H(P)$. Let \mathcal{F} be a family of disjoint subsets of sizes $\{m_1, \dots, m_s\}$. We define $M_{\mathcal{F}} \triangleq \sum_{i=1}^s m_i$, and $Q_{\mathcal{F}}$ to be the distribution induced by the family, namely the distribution $(m_1/M_{\mathcal{F}}, \dots, m_s/M_{\mathcal{F}})$.

The Shannon capacity satisfies [1]

$$\log \alpha(G) \leq C(G) \leq \log \text{cc}(G) \quad (5)$$

where $\alpha(G)$ is the independence number of G and $\text{cc}(G)$ is the vertex clique covering number of G . A *vertex clique covering* of a graph G is set of cliques such that every vertex of G is a member of exactly one clique. A minimum clique covering is a clique covering of minimum size, and the *clique covering number* $\text{cc}(G)$ is the size of a minimum clique covering. Now we describe a natural generalization of this bound to the ρ -capacity, which also include the bound (5) as a special case when $\rho = 0$. Note that the derivations in the proofs of Theorems 2–3 are incidentally very similar to those of [12, Chapter 5], where Jelinek calculated the error exponents in source coding.

Theorem 2. *Let G be a graph with m vertices. Suppose that \mathcal{F}_1 is an independent family of G and \mathcal{F}_2 is a vertex clique cover of G . Then*

$$C_\rho(G) \geq \inf_{\beta \in [0,1]} (1-\beta)H_\beta(Q_{\mathcal{F}_1}) + \beta(\log M_{\mathcal{F}_1} - \rho) \quad (6)$$

for $0 \leq \rho \leq \log M_{\mathcal{F}_1}$, and

$$C_\rho(G) \leq \inf_{\beta \in [0,1]} (1-\beta)H_\beta(Q_{\mathcal{F}_2}) + \beta(\log m - \rho) \quad (7)$$

for $0 \leq \rho \leq \log m$. Moreover, if G is a disjoint union of cliques, then both bounds coincide (and are hence tight). The minimizing β is given in (12) and (15).

Proof: We prove the bounds are tight for $G = K_{m_1} + K_{m_2} + \dots + K_{m_s}$ that is a disjoint union of cliques.³ The lower bound for a general graph follows since given an independent family, we can consider the associated induced subgraph, and add edges to create a disjoint union of cliques (hence decrease the ρ -capacity). The upper bound for a general graph will follow by noting that given a vertex clique cover, we can remove edges (hence increase the ρ -capacity) to create a disjoint union of cliques.

The case that $m_i, 1 \leq i \leq s$ are all equal is easy to prove, thus we can assume that $s \geq 2$ and $m_i, 1 \leq i \leq s$ are not all equal. Set $\mathcal{F} = \{V(K_{m_1}), \dots, V(K_{m_s})\}$ and $m = m_1 + \dots + m_s$. Now we have

$$G^n = (K_{m_1} + K_{m_2} + \dots + K_{m_s})^n \cong \sum_{i_1 + \dots + i_s = n} \binom{n}{i_1, i_2, \dots, i_s} K_{m_1^{i_1} \dots m_s^{i_s}}, \quad (8)$$

where the sum is taken over all combinations of nonnegative integer indices i_1 through i_s such that the sum of all i_j is n . From (8) we see G^n is also a disjoint union of cliques. Let us write G^n as a disjoint union of small and large cliques, i.e., $G^n = G_1 + G_2$ where

$$G_1 = \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} < 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s} K_{m_1^{i_1} \dots m_s^{i_s}},$$

$$G_2 = \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} \geq 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s} K_{m_1^{i_1} \dots m_s^{i_s}}.$$

³Note that the bounds for a disjoint union of cliques appear implicitly in [4], [5], [6], [7]. Here we provide the exact analytical expression.

It is easy to see that $\alpha_{2^{\rho n}}(G_2) \leq \alpha_{2^{\rho n}}(G^n) \leq \alpha_{2^{\rho n}}(G_1) + \alpha_{2^{\rho n}}(G_2)$, and

$$\begin{aligned}\alpha_{2^{\rho n}}(G_1) &\leq A(n) \triangleq \frac{1}{2^{\rho n}} \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} \leq 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s} m_1^{i_1} \dots m_s^{i_s} = \frac{|V(G_1)|}{2^{\rho n}}, \\ \alpha_{2^{\rho n}}(G_2) &= B(n) \triangleq \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} \geq 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s}.\end{aligned}$$

Hence

$$B(n) \leq \alpha_{2^{\rho n}}(G^n) \leq A(n) + B(n). \quad (9)$$

Suppose that

$$\frac{1}{s} \sum_{i=1}^s \log m_i \leq \rho \leq \frac{1}{m} \sum_{i=1}^s m_i \log m_i. \quad (10)$$

By (9) and Lemma 8 of the Appendix, we have

$$C_\rho(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log A(n) = \lim_{n \rightarrow \infty} \frac{1}{n} \log B(n) = \log \left(\sum_{i=1}^s m_i^\beta \right) - \beta \rho = (1 - \beta) H_\beta(Q_{\mathcal{F}}) + \beta (\log M_{\mathcal{F}} - \rho) \quad (11)$$

where $\beta \in [0, 1]$ is the unique solution such that

$$\rho = \left(\sum_{i=1}^s m_i^\beta \log m_i \right) / \sum_{i=1}^s m_i^\beta. \quad (12)$$

Note that: if $\rho = \frac{1}{s} \sum_{i=1}^s \log m_i$ associated with $\beta = 0$ then $C_\rho(G) = C(G) = \log s$; and if $\rho = \frac{1}{m} \sum_{i=1}^s m_i \log m_i$ associated with $\beta = 1$ then $C_\rho(G) = \log m - \rho$. Therefore

$$C_\rho(G) = \begin{cases} \log s & \text{if } 0 \leq \rho \leq \frac{1}{s} \sum_{i=1}^s \log m_i \\ \log m - \rho & \text{if } \frac{1}{m} \sum_{i=1}^s m_i \log m_i \leq \rho \leq \log m, \end{cases} \quad (13)$$

where the first equality follows from the monotonically decreasing property of $C_\rho(G)$, and the second follows by time sharing.

Via direct computation, we can verify

$$\begin{aligned}C_\rho(G) &= \inf_{\beta \in [0, 1]} \log \left(\sum_{i=1}^s m_i^\beta \right) - \beta \rho \\ &= \inf_{\beta \in [0, 1]} (1 - \beta) H_\beta(Q_{\mathcal{F}}) + \beta (\log M_{\mathcal{F}} - \rho)\end{aligned} \quad (14)$$

for $0 \leq \rho \leq \log M_{\mathcal{F}}$. Here

$$\begin{aligned}\beta^* &= \arg \inf_{\beta \in [0, 1]} (1 - \beta) H_\beta(Q_{\mathcal{F}}) + \beta (\log M_{\mathcal{F}} - \rho) \\ &= \begin{cases} 0 & \text{if } 0 \leq \rho \leq \frac{1}{s} \sum_{i=1}^s \log m_i \\ 1 & \text{if } \frac{1}{m} \sum_{i=1}^s m_i \log m_i \leq \rho \leq \log M_{\mathcal{F}}, \end{cases}\end{aligned} \quad (15)$$

and $\beta^* \in [0, 1]$ is the unique solution satisfying (12) when ρ satisfies (10). ■

Remark 1. Note that for $\rho = 0$, the bound (6) yields $C(G) \geq \log \alpha(G)$. Moreover, if we pick $\beta = 0$ in (7), then it follows that $C_\rho(G) \leq \log cc(G)$, and if we pick $\beta = 1$ in (7), then it follows that $C_\rho(G) \leq \log m - \rho$.

In the following theorem we provide an alternative characterization for the ρ -capacity of a disjoint union of cliques, via its concave conjugate. We also explicitly find the associated free-lunch point and packing point.

Theorem 3. Let $G = K_{m_1} + K_{m_2} + \dots + K_{m_s}$ be a disjoint union of cliques and $m = m_1 + \dots + m_s$. Suppose that $s \geq 2$ and $m_i, 1 \leq i \leq s$ are not all equal.

1) The concave conjugate of the ρ -capacity is given by

$$C_\star(G, \gamma) = -\log \sum_{i=1}^s m_i^{-\gamma} \quad \text{for } -1 \leq \gamma \leq 0.$$

2) The ρ -capacity $C_\rho(G)$ is differentiable on $[0, \log m]$ and

$$C'_\rho(G) = -\arg \inf_{\beta \in [0,1]} (1-\beta)H_\beta(Q_{\mathcal{F}}) + \beta(\log M_{\mathcal{F}} - \rho).$$

3) The free-lunch point $\rho^*(G) = \frac{1}{s} \sum_{i=1}^s \log m_i$.

4) The packing point $\rho_*(G) = \frac{1}{m} \sum_{i=1}^s m_i \log m_i$.

Proof: Let $g(\gamma) = -\log \sum_{i=1}^s m_i^{-\gamma}$ for $\gamma \in [-1, 0]$. Define $g_\star : [0, \log m] \rightarrow \mathbb{R}$ to be the concave conjugate of g , namely $g_\star(\rho) = \inf_{\gamma \in [-1,0]} \rho\gamma - g(\gamma)$. By (14) we have

$$\begin{aligned} C_\rho(G) &= \inf_{\beta \in [0,1]} \log \left(\sum_{i=1}^s m_i^\beta \right) - \beta\rho \\ &= \inf_{\gamma \in [-1,0]} \rho\gamma + \log \sum_{i=1}^s m_i^{-\gamma} \\ &= g_\star(\rho). \end{aligned}$$

By the Fenchel–Moreau Theorem [13, Exercise 3.39] and [14, Theorem 4.1.1], we have $C_\star(G, \gamma) = g(\gamma)$ for $-1 \leq \gamma \leq 0$, and

$$C'_\rho(G) = g'_\star(\rho) = \arg \inf_{\gamma \in [-1,0]} \gamma\rho - g(\gamma) = -\arg \inf_{\beta \in [0,1]} (1-\beta)H_\beta(Q_{\mathcal{F}}) + \beta(\log M_{\mathcal{F}} - \rho).$$

This proves 1) and 2). Thus, we have $C'_\rho(G) < 0$ for $\rho > \frac{1}{s} \sum_{i=1}^s \log m_i$ and $C'_\rho(G) > -1$ for $\rho < \frac{1}{m} \sum_{i=1}^s m_i \log m_i$, and then 3) and 4) follow. \blacksquare

Next, we provide bounds on the free-lunch point and packing point of general graphs.

Corollary 2. *Let G be a graph with m vertices.*

- 1) *Suppose that G has s connected components of sizes m_1, \dots, m_s . Then the packing point satisfies $\rho_*(G) \leq \frac{1}{m} \sum_{i=1}^s m_i \log m_i$.*
- 2) *Let t be a positive integer, and let $\mathcal{F} = \{V_1, \dots, V_n\}$ be an independent family of G^t . If $C_0(G) = (\log |\mathcal{F}|)/t$ (the Shannon capacity is finitely attained), then the free-lunch point satisfies $\rho^*(G) \geq \frac{1}{tn} \sum_{i=1}^n \log |V_i|$.*

Proof:

- 1) Note that these connected components trivially form an independent family of G . From (6) and (13), we have $C_\rho(G) \geq \log m - \rho$ for $\frac{1}{m} \sum_{i=1}^s m_i \log m_i \leq \rho \leq \log m$. This meets the upper bound of Proposition 5, and is hence tight.
- 2) By (6) and (13) we have $C_\rho(G) = C_0(G) = (\log |\mathcal{F}|)/t$ for $0 \leq \rho \leq \frac{1}{tn} \sum_{i=1}^n \log |V_i|$. Hence $\rho^*(G) \geq \frac{1}{tn} \sum_{i=1}^n \log |V_i|$. \blacksquare

We will later prove (in Theorem 10) that the upper bound on $\rho_*(G)$ in Corollary 2 is in fact always attained.

Example 4. Let G be the graph with vertex set $\{1, 2, 3, 4, 5\}$ and edge set $\{12, 13, 23, 34, 45, 15\}$ (pentagon with an extra edge). It is easy to see that $C_0(G) = 1$. Using the independent family $\mathcal{F} = \{\{2\}, \{4, 5\}\}$ in Corollary 2, we see that G has a nontrivial free-lunch point $\rho^*(G) \geq 1/2 > 0$.

Example 5. Let $K_{m,n}$ ($m \geq n \geq 2$) be the complete bipartite graph whose vertices can be partitioned into two subsets $V_1 = \{1, 2, \dots, m\}$ and $V_2 = \{m+1, \dots, m+n\}$ such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. Let G be obtained from $K_{m,n}$ by deleting the edges that connect $m+1$ and $2, 3, \dots, m$. Using the independent family $\mathcal{F} = \{\{1, m+1\}, \{2\}, \{3\}, \dots, \{m\}\}$ in Corollary 2 we see that G has a nontrivial free-lunch point $\rho^*(G) \geq 1/m > 0$.

C. A Lovász-Type Upper Bound for Regular Graphs

In this section, we give an upper bound for the ρ -capacity of regular graphs. Our approach follows the technique developed in [8], [9], which generalized Lovász's brilliant idea [2].

Let $G = (V, E)$ be a graph with m vertices and $e \geq 1$ edges. The graph G is said to be *regular* if the number of edges containing a given vertex v is a constant r , independent of v , called the *degree* of G . Let B be the adjacency matrix of G , and μ be its smallest eigenvalue. It is well-known that $\mu \leq -1$ if G has at least one edge. Set $A = I + |\mu|^{-1}B$. For two matrices M and N , we use $M \otimes N$ to denote their Kronecker product. The matrix $M^{\otimes n}$ is defined inductively by $M^{\otimes n} \triangleq M^{\otimes n-1} \otimes M$. Define

$$\lambda(M) \triangleq \inf\{\mathbf{x}^T M \mathbf{x} : \sum \mathbf{x}(i) = 1\}.$$

Lemma 2 ([2], [8]).

- 1) $\lambda(A^{\otimes n}) = \lambda(A)^n$.
- 2) $C(G) \leq \log \lambda(A)^{-1} = \log \frac{m|\mu|}{r+|\mu|}$.

Proof: See Section II of [8]. ■

Set $V = \{v_1, \dots, v_m\}$. Let \mathcal{F} be a k -independent family in G . We can assume that each subset in \mathcal{F} contains exactly k vertices; otherwise we can form another k -independent family \mathcal{F}' by choosing exactly k vertices from each subset. For a vertex $v \in V$, we say that $v \in \mathcal{F}$ if v is contained in some subset of \mathcal{F} . Recall that we write $|\mathcal{F}|$ for the number of subsets in \mathcal{F} . We now define a length m vector \mathbf{y} by $\mathbf{y}(i) = 1/(k|\mathcal{F}|)$ if $v_i \in \mathcal{F}$; otherwise $\mathbf{y}(i) = 0$. Then $\sum_i \mathbf{y}(i) = 1$ and

$$\lambda(A) \leq \mathbf{y}^T A \mathbf{y} = \frac{1}{(k|\mathcal{F}|)^2} (k|\mathcal{F}| + \sum_{\substack{v_i, v_j \in \mathcal{F} \\ v_i \sim v_j}} A(i, j)). \quad (16)$$

Let \mathcal{F}_n be a maximum $2^{\rho n}$ -independent family of graph G^n . Then by Lemma 2 and (16) we obtain

$$\lambda(A)^n = \lambda(A^{\otimes n}) \leq \frac{2^{\rho n} |\mathcal{F}_n| + \sum_{i=1}^n s_i^{(n)} |\mu|^{-i}}{(2^{\rho n} |\mathcal{F}_n|)^2}, \quad (17)$$

where $s_i^{(n)}$ is the number of pairs $(\mathbf{u}, \mathbf{v}) \in \mathcal{F}_n \times \mathcal{F}_n$ such that $A^{\otimes n}(\mathbf{u}, \mathbf{v}) = |\mu|^{-i}$. We now give an upper bound for $s_i^{(n)}$ and the sum $\sum_{i=1}^n s_i^{(n)} |\mu|^{-i}$ through a simple counting argument. First let us introduce two functions which will simplify our derivations. Recall that m is the number of vertices of G and e is its number of edges. For $1 \leq i \leq n$, let

$$f(i) \triangleq m^{n-i} (2e)^i \binom{n}{i} \quad \text{and} \quad g(i) \triangleq m^{n-i} (2e)^i \binom{n}{i} |\mu|^{-i}.$$

Lemma 3.

- 1) For any $1 \leq i \leq n$,

$$s_i^{(n)} \leq f(i),$$

$$\sum_{i=1}^n s_i^{(n)} \leq 2^{\rho n} (2^{\rho n} - 1) |\mathcal{F}_n|.$$

- 2) For any $1 \leq k \leq n$,

$$\sum_{i=1}^n s_i^{(n)} |\mu|^{-i} \leq 2^{\rho n} (2^{\rho n} - 1) |\mathcal{F}_n| |\mu|^{-k} + \sum_{i=1}^k g(i).$$

- 3) For any $0 \leq q \leq 1$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log f(qn) = \log(m(r+1)) - D\left(q \parallel \frac{r}{r+1}\right),$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log g(qn) = \log \frac{m(r+|\mu|)}{|\mu|} - D\left(q \parallel \frac{r}{r+|\mu|}\right).$$

4) Write $C_\rho = C_\rho(G)$. Then for $0 \leq q \leq \frac{r}{r+|\mu|}$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^n s_i^{(n)} |\mu|^{-i} \\ & \leq \max \left\{ 2\rho + C_\rho - q \log |\mu|, \right. \\ & \quad \left. \log \frac{m(r+|\mu|)}{|\mu|} - D \left(q \parallel \frac{r}{r+|\mu|} \right) \right\}. \end{aligned}$$

Proof:

1) For two vertices $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ of G^n , the element $A^{\otimes n}(\mathbf{u}, \mathbf{v}) = |\mu|^{-i}$ if and only if there are $n-i$ pairs of coordinates such that $u_j = v_j$ and the other i pairs are adjacent in graph G . So for $1 \leq i \leq n$ we get

$$s_i^{(n)} \leq m^{n-i} (2e)^i \binom{n}{i} = f(i).$$

We see that $\frac{1}{2} \sum_{i=1}^n s_i^{(n)}$ is the number of edges that connect pairs of vertices in \mathcal{F}_n . By the definition of a $2^{\rho n}$ -independent family, this number is upper bounded by $\frac{1}{2} 2^{\rho n} (2^{\rho n} - 1) |\mathcal{F}_n|$.

2) We have

$$\begin{aligned} \sum_{i=1}^n s_i^{(n)} |\mu|^{-i} & \leq \sum_{i=1}^k g(i) + \sum_{k+1}^n s_i^{(n)} |\mu|^{-i} \\ & \leq \sum_{i=1}^k g(i) + |\mu|^{-k} \sum_{k+1}^n s_i^{(n)} \\ & \leq \sum_{i=1}^k g(i) + |\mu|^{-k} 2^{\rho n} (2^{\rho n} - 1) |\mathcal{F}_n|. \end{aligned}$$

3) Note that the first equality is a special case of the second with $|\mu| = 1$. We have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log g(qn) \\ & = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(m^{n-qn} (2e)^{qn} \binom{n}{qn} |\mu|^{-qn} \right) \\ & = (1-q) \log m + q \log (2e) + h(q) - q \log |\mu| \\ & = \log \frac{m(r+|\mu|)}{|\mu|} - D \left(q \parallel \frac{r}{r+|\mu|} \right). \end{aligned}$$

4) Set $k = qn$ in 2), and choose \mathcal{F}_n to asymptotically achieve C_ρ . Then this argument follows from

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log (2^{\rho n} (2^{\rho n} - 1) |\mathcal{F}_n| |\mu|^{-qn}) \\ & = 2\rho + C_\rho - q \log |\mu|, \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^{qn} g(i) \\ & = \log \frac{m(r+|\mu|)}{|\mu|} - D \left(q \parallel \frac{r}{r+|\mu|} \right). \end{aligned}$$

■

We are now ready to state our bound.

Theorem 4. Let $G = (V, E)$ be a regular graph with m vertices, e edges and degree r . Let μ be its smallest eigenvalue. Then for any ρ satisfying

$$\frac{1}{2} \log \frac{r+|\mu|}{|\mu|} < \rho < \log \frac{r+|\mu|}{|\mu|} + \frac{r}{r+|\mu|} \log |\mu|, \quad (18)$$

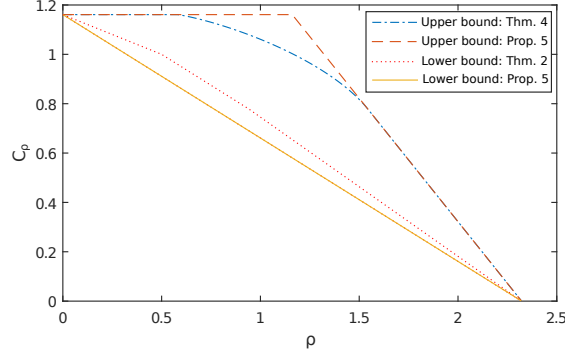


Fig. 1. Bounds on the ρ -capacity of the Pentagon (Example 6)

it holds that

$$C_\rho(G) \leq \log m - \rho - \frac{1}{2}D \left(p \parallel \frac{r}{r + |\mu|} \right), \quad (19)$$

where $0 < p < \frac{r}{r + |\mu|}$ is the unique solution of

$$\rho = \log \frac{r + |\mu|}{|\mu|} + p \log |\mu| - \frac{1}{2}D \left(p \parallel \frac{r}{r + |\mu|} \right). \quad (20)$$

Proof: Write $C_\rho = C_\rho(G)$, and let \mathcal{F}_n asymptotically achieve C_ρ . Suppose $0 < q < \frac{r}{r + |\mu|}$. By claim 2) of Lemma 2, inequality (17) and claim 4) of Lemma 3, we have

$$\begin{aligned} -\log \frac{m|\mu|}{r + |\mu|} &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{2^{\rho n} |\mathcal{F}_n| + \sum_{i=1}^n s_i^{(n)} |\mu|^{-i}}{(2^{\rho n} |\mathcal{F}_n|)^2} \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(2^{\rho n} |\mathcal{F}_n| + \sum_{i=1}^n s_i^{(n)} |\mu|^{-i} \right) - 2\rho - 2C_\rho \\ &\leq \max \left\{ C_\rho + \rho, 2\rho + C_\rho - q \log |\mu|, \log \frac{m(r + |\mu|)}{|\mu|} - D \left(q \parallel \frac{r}{r + |\mu|} \right) \right\} - 2\rho - 2C_\rho. \end{aligned} \quad (21)$$

Rearranging the terms in (21) we get

$$C_\rho \leq \max \left\{ \log \frac{m|\mu|}{r + |\mu|} - \rho, \log \frac{m|\mu|}{r + |\mu|} - q \log |\mu|, \log m - \rho - \frac{1}{2}D \left(q \parallel \frac{r}{r + |\mu|} \right) \right\}. \quad (22)$$

The difference of the last two terms in (22) is

$$\begin{aligned} \Delta(q) &\triangleq \left(\log \frac{m|\mu|}{r + |\mu|} - q \log |\mu| \right) - \left(\log m - \rho - \frac{1}{2}D \left(q \parallel \frac{r}{r + |\mu|} \right) \right) \\ &= \rho - \left(\log \frac{r + |\mu|}{|\mu|} + q \log |\mu| - \frac{1}{2}D \left(q \parallel \frac{r}{r + |\mu|} \right) \right). \end{aligned}$$

It is easy to check that $\Delta(q)$ is a continuous and strictly decreasing function of q in the interval $[0, \frac{r}{r + |\mu|}]$. Moreover, for any ρ satisfying (18), $\Delta(0) > 0$ and $\Delta(r/(r + |\mu|)) < 0$. Hence there exists a unique value p satisfying (20), and the bound (19) follows by setting $q = p$, which is optimal, in (22). ■

Example 6. We apply Theorem 4 to the cycle graph C_5 (pentagon), whose vertex set is $\{1, 2, 3, 4, 5\}$ and edge set is $\{12, 23, 34, 45, 51\}$. Then $r = 2$ and $\mu = -(\sqrt{5} + 1)/2$. The results are depicted in Figure 1. A lower bound can be obtained using (6) of Theorem 2 with the following three independent families of C_5^2 : (i) $\{\{1\} \times \{1, 2, 3, 4, 5\}, \{3\} \times \{1, 2, 3, 4, 5\}, \{4\} \times \{1, 2, 3, 4, 5\}\}$; (ii) $\{\{4\} \times \{1, 2\}, \{1, 2\} \times \{1, 2\}, \{1, 2, 3, 4, 5\} \times \{4\}\}$; (iii) $\{\{4, 5\} \times \{5\}, \{2\} \times \{1, 5\}, \{1, 2\} \times \{3\}, \{4\} \times \{2, 3\}\}$. The other bounds are obtained using Proposition 5.

IV. SOME PROPERTIES OF THE ρ -CAPACITY

In this section, we study the properties of the ρ -capacity function under two graph operations: the strong product and the disjoint union. To that end, we first observe the following three simple lemmas, whose proofs are relegated to the Appendix.

Lemma 4. *Let G be a graph with m vertices and $2 \leq k \leq m$. Suppose $\mathcal{F} = \{V_i \mid 1 \leq i \leq N\}$ is an independent family of G such that $|V_i| \leq k$ for $1 \leq i \leq N$. Then*

$$\sum_{i=1}^N |V_i| \leq \min\{m, (k-1)(2\alpha_k(G) + 1)\}.$$

Lemma 5. *Let $G = H_1 + H_2 + \dots + H_n$ be the disjoint union of n graphs H_1, \dots, H_n and $k \geq 2$. Then*

$$\sum_{i=1}^n \alpha_k(H_i) \leq \alpha_k(G) \leq \min\left\{\frac{|V(G)|}{k}, \frac{k-1}{k} \sum_{i=1}^n (2\alpha_k(H_i) + 1)\right\}.$$

Lemma 6. *Let G be a graph. Then for any positive integer $k \leq |V(G)|$, we have $\alpha_k(G) = \alpha_{km}(G \boxtimes K_m)$.*

A. Strong Product

The following theorem provides a lower bound on the ρ -capacity of a strong product.

Theorem 5. *Let G be a graph with m_1 vertices and H a graph with m_2 vertices. Then*

$$C_\rho(G \boxtimes H) \geq \max_{\rho_1 + \rho_2 = \rho} C_{\rho_1}(G) + C_{\rho_2}(H) \quad (23)$$

where the max is taken over $0 \leq \rho_1 \leq \log m_1$ and $0 \leq \rho_2 \leq \log m_2$ such that $\rho_1 + \rho_2 = \rho$.

Proof: By Lemma 1 we get

$$\alpha_{2(\rho_1 + \rho_2)n}((G \boxtimes H)^n) \geq \alpha_{2\rho_1 n}(G^n) \cdot \alpha_{2\rho_2 n}(H^n).$$

The result immediately follows. ■

Remark 2. We note that Theorem 5 says that the function $C_\rho(G \boxtimes H)$ is lower bounded by the *supremal convolution* of $C_\rho(G)$ and $C_\rho(H)$. Equivalently, it says that the hypograph of $C_\rho(G \boxtimes H)$ contains the Minkowsky sum of the hypographs of $C_\rho(G)$ and $C_\rho(H)$.

The following is a simple corollary of Theorem 5.

Corollary 3. *The concave conjugate of the ρ -capacity is subadditive with respect to the strong product, i.e.,*

$$C_\star(G \boxtimes H, \gamma) \leq C_\star(G, \gamma) + C_\star(H, \gamma).$$

Proof: Let $m_1 = |V(G_1)|$ and $m_2 = |V(G_2)|$. Then

$$\begin{aligned} C_\star(G \boxtimes H, \gamma) &= \inf_{\rho \in [0, \log(m_1 m_2)]} \gamma \rho - C_\rho(G \boxtimes H) \\ &= \inf_{\substack{\rho_1 \in [0, \log m_1] \\ \rho_2 \in [0, \log m_2]}} \gamma(\rho_1 + \rho_2) - C_{\rho_1 + \rho_2}(G \boxtimes H) \\ &\leq \inf_{\substack{\rho_1 \in [0, \log m_1] \\ \rho_2 \in [0, \log m_2]}} \gamma(\rho_1 + \rho_2) - C_{\rho_1}(G) - C_{\rho_2}(H) \\ &= \left(\inf_{\rho_1 \in [0, \log m_1]} \gamma \rho_1 - C_{\rho_1}(G) \right) + \left(\inf_{\rho_2 \in [0, \log m_2]} \gamma \rho_2 - C_{\rho_2}(H) \right) \\ &= C_\star(G, \gamma) + C_\star(H, \gamma), \end{aligned}$$

where the inequality follows from Theorem 5. ■

When H is taken to be a complete graph in Theorem 5, then the lower bound (23) is attained.

Theorem 6. For any graph G , we have

$$C_\rho(G \boxtimes K_m) = \begin{cases} C(G) & \text{if } 0 \leq \rho < \log m \\ C_{\rho - \log m}(G) & \text{if } \log m \leq \rho \leq \log |V(G \boxtimes K_m)|. \end{cases}$$

Proof: First we can readily verify that $C(G \boxtimes K_m) = C(G)$. Since $(G \boxtimes K_m)^n \cong G^n \boxtimes K_{m^n}$, by Lemma 6 we have

$$\alpha_{2^{\rho n}}(G^n) = \alpha_{2^{\rho n} m^n}(G^n \boxtimes K_{m^n}).$$

It follows that

$$C_\rho(G \boxtimes K_m) = C_{\rho - \log m}(G) \text{ for } \rho \geq \log m.$$

For $0 < \rho < \log m$, the result follows from the monotonically decreasing property of the ρ -capacity. ■

B. Disjoint Union

We begin by finding the ρ -capacity of a union of two identical graphs, in terms of the ρ -capacity of a single copy.

Theorem 7. Let G be a graph with m vertices. Then

$$C_\rho(G + G) = \begin{cases} 1 + C_\rho(G) & \text{if } 0 \leq \rho < \log m \\ 1 - \rho + \log m & \text{if } \log m \leq \rho \leq 1 + \log m. \end{cases}$$

Proof: It can be easily verified that $C(G + G) = 1 + C(G)$. Now we consider the case $0 < \rho \leq \log m$. From $(G + G)^n \cong 2^n G^n$ and Lemma 5 we have

$$2^n \cdot \alpha_{2^{\rho n}}(G^n) \leq \alpha_{2^{\rho n}}((G + G)^n) < 2^n \cdot (2 \alpha_{2^{\rho n}}(G^n) + 1).$$

It follows that $C_\rho(G + G) = 1 + C_\rho(G)$ for $0 < \rho \leq \log m$. The remaining case follows directly from 1) of Corollary 2. ■

We now provide a lower bound on the ρ -capacity of a general disjoint union.

Theorem 8. Let G be a graph with m_1 vertices and H a graph with m_2 vertices, and let

$$\delta = (m_1 \log m_1 + m_2 \log m_2) / (m_1 + m_2).$$

Then

$$C_\rho(G + H) \begin{cases} \geq \max_{p\rho_1 + (1-p)\rho_2 = \rho} h(p) + p C_{\rho_1}(G) + (1-p) C_{\rho_2}(H) & \text{if } 0 \leq \rho < \delta \\ = \log(m_1 + m_2) - \rho & \text{if } \delta \leq \rho \leq \log(m_1 + m_2), \end{cases} \quad (24)$$

where the max is taken over $0 \leq p \leq 1, 0 \leq \rho_1 \leq \log m_1$, and $0 \leq \rho_2 \leq \log m_2$ such that $p\rho_1 + (1-p)\rho_2 = \rho$.

Proof: By 1) of Corollary 2 we have $C_\rho(G + H) = \log(m_1 + m_2) - \rho$ if $\rho \geq \frac{1}{m_1 + m_2} \sum_{i=1}^2 m_i \log m_i = \delta$. By the properties of the strong product we have

$$(G + H)^n \cong \sum_{i=0}^n \binom{n}{i} G^i \boxtimes H^{n-i}.$$

Let $i = pn$. By Lemma 1 we have

$$\alpha_{2^{p\rho_1 + (1-p)\rho_2}n}((G + H)^n) \geq \binom{n}{pn} \cdot \alpha_{2^{p\rho_1}n}(G^{pn}) \cdot \alpha_{2^{p\rho_2(1-p)n}}(H^{(1-p)n}).$$

Now the first inequality follows directly. ■

If H is taken to be a complete graph in Theorem 8, then the lower bound (24) is attained.

Theorem 9. Let G be a graph with m_1 vertices, and let

$$\delta = (m_1 \log m_1 + m_2 \log m_2) / (m_1 + m_2).$$

Then

$$C_\rho(G + K_{m_2}) = \begin{cases} \max_{p\rho_1 + (1-p)\log m_2 \geq \rho} h(p) + p C_{\rho_1}(G) & \text{if } 0 \leq \rho < \delta \\ \log(m_1 + m_2) - \rho & \text{if } \delta \leq \rho \leq \log m_1 + m_2. \end{cases}$$

Here the max is taken over $0 \leq p \leq 1$ and $0 \leq \rho_1 \leq \log m_1$ such that $p\rho_1 + (1-p)\log m_2 \geq \rho$.

Proof: The second equality follows from 1) of Corollary 2 directly. We have

$$(G + K_{m_2})^n \cong \sum_{i=0}^n \binom{n}{i} G^i \boxtimes K_{m_2^{n-i}}.$$

First, assume that $m_1 = m_2$ and $0 \leq \rho < \delta = \log m_1$. Then by Lemma 5 we have

$$\sum_{i=0}^n \binom{n}{i} \alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}) \leq \alpha_{2^{\rho n}}((G + K_{m_2})^n) < \sum_{i=0}^n \binom{n}{i} (2 \alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}) + 1). \quad (25)$$

Since $\alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}) \geq 1$, we get

$$C_\rho(G + K_{m_2}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i=0}^n \binom{n}{i} \alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}) \right). \quad (26)$$

Let $i = pn$, and $\rho_1 = (\rho - (1-p)\log m_2)/p$. Through a similar analysis as Theorem 6, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_{2^{\rho n}}(G^{pn} \boxtimes K_{m_2^{(1-p)n}}) = \begin{cases} p C_{\rho_1}(G) & \text{if } \rho \geq (1-p)\log m_2 \\ p C_0(G) & \text{if } \rho < (1-p)\log m_2. \end{cases} \quad (27)$$

Combining this with (26) we get

$$C_\rho(G + K_{m_2}) = \max_{p\rho_1 + (1-p)\log m_2 \geq \rho} h(p) + p C_{\rho_1}(G).$$

Secondly, assume that $m_1 < m_2$. If $0 \leq \rho \leq \log m_1$, then it can be proved similarly as above. Thus, we can assume that $\log m_1 < \rho < \delta$. Let N be the largest integer such that $|V(G^i \boxtimes K_{m_2^{n-i}})| = m_1^N m_2^{n-N} \geq 2^{\rho n}$. Set

$$\begin{aligned} A(n) &= \sum_{i=0}^N \binom{n}{i} \alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}), \\ B(n) &= \frac{1}{2^{\rho n}} \sum_{i=N}^n \binom{n}{i} m_1^i m_2^{n-i}. \end{aligned}$$

Then

$$A(n) \leq \alpha_{2^{\rho n}}((G + K_{m_2})^n). \quad (28)$$

By Lemma 5 we have

$$\begin{aligned} \alpha_{2^{\rho n}}((G + K_{m_2})^n) &< \sum_{i=0}^N \binom{n}{i} (2 \alpha_{2^{\rho n}}(G^i \boxtimes K_{m_2^{n-i}}) + 1) + 2 \left(\alpha_{2^{\rho n}} \left(\sum_{i=N}^n \binom{n}{i} G^i \boxtimes K_{m_2^{n-i}} \right) + 1 \right) \\ &\leq 2(A(n) + B(n)) + \sum_{i=0}^N \binom{n}{i} + 2. \end{aligned} \quad (29)$$

Following a similar analysis as above, we can obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log A(n) = \max_{p\rho_1 + (1-p)\log m_2 \geq \rho} h(p) + p C_{\rho_1}(G). \quad (30)$$

Following the same method of Lemma 8 of the Appendix, we can prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log B(n) = h(q) \quad (31)$$

where $q \in [0, 1]$ satisfying that $q \log m_1 + (1 - q) \log m_2 = \rho$. Combining (28) – (31), we get

$$\begin{aligned} C_\rho(G) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log A(n) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(2(A(n) + B(n)) + \sum_{i=0}^N \binom{n}{i} + 2 \right) \\ &= \max_{p\rho_1 + (1-p)\log m_2 \geq \rho} h(p) + p C_{\rho_1}(G). \end{aligned}$$

The case $m_1 > m_2$ can be proved similarly. ■

Remark 3. Let G and H be two graphs. Suppose that $\Theta(G) = \log A$ and $\Theta(H) = \log B$ for some $A > 0$ and $B > 0$. Shannon [1, Theorem 4] proved that $\Theta(G + H) \geq \log(A + B)$ and $\Theta(G \boxtimes H) \geq \log A + \log B$, and that both bounds hold with equality if the vertex set of one of the two graphs, say G , can be covered by $\alpha(G)$ cliques. He also conjectured that the equalities hold in general, which has been disproved by Alon [15]. Theorems 8–9 can be seen as generalizations of [1, Theorem 4] to the ρ -capacity setting. It is thus interesting to ask whether (24) holds with equality under Shannon’s conditions.

V. ρ -CAPACITY AND STRUCTURAL PROPERTIES

In this section, we discuss the connection between the ρ -capacity and some simple structural properties of the graph. First, we prove that a graph G is not connected if and only if its packing point $\rho_*(G) < \log |V(G)|$. More explicitly, we show that the gap between $\rho_*(G)$ and $\log |V(G)|$ is equal to the Shannon entropy of the distribution induced by the sizes of the connected components of G . We then proceed to show that the free-lunch point and the packing point of a graph G coincide if and only if $G = sK_n$ for some positive integers s and n . Lastly, we show that two disjoint union of cliques are isomorphic if and only if their ρ -capacity functions are the same.

Theorem 10. *Let G be a graph with m vertices. Suppose that G has s connected components of sizes m_1, \dots, m_s . Let $Q = \{m_1/m, \dots, m_s/m\}$. Then $\rho_*(G) = \log m - H(Q)$. In particular, $\rho_*(G) < \log m$ if and only if G is not connected.*

Proof: Let G be the confusion graph of some point-to-point channel $p(y|x)$. By Theorem 1 we have

$$\begin{aligned} C_\rho(G) &\leq \max_{\substack{U-X-Y \\ H(X|U) \geq \rho}} I(Y; U) \\ &= \max_{\substack{U-X-Y \\ H(X|U) \geq \rho}} H(X) - I(U; X|Y) - H(X|U) \\ &\leq \log m - \rho. \end{aligned}$$

Thus, a necessary condition to achieve a sum-rate of $\log m$ is that $H(X|U) = \rho$, $H(X) = \log m$ (i.e., X is uniform), and $I(U; X|Y) = 0$ (i.e., $U - Y - X$ also forms a Markov chain). Hence we can lower bound the packing point by

$$\rho_*(G) \geq \min_{\substack{U-X-Y \\ U-Y-X \\ H(X) = \log m}} H(X|U).$$

For every y , define $S_y = \{x \mid x \in \mathcal{X}, p(x|y) > 0\}$. Clearly, S_y is a clique in G . The two Markov chains imply that $p(u|x) = p(u|y)$ whenever $p(u, x, y) > 0$. Hence for any y , the distribution $p(u|x)$ is the same for each $x \in S_y$.

This immediately implies that if G is connected then $p(u|x)$ does not depend on x at all. Hence U and X are independent, and thus

$$\rho_*(G) \geq \min_{\substack{U: U-X-Y \\ U-Y-X \\ H(X) = \log m}} H(X|U) = H(X) = \log m.$$

Now assume that G is not connected. From the above arguments it is clear that $p(u|x)$ does not change inside each connected component of G . In other words, we have the Markov chain $U - Z - X$ where $Z = g(X)$ is a random variable that returns the index of the connected component of G that X lies in. Then $H(X|U) \geq H(X|Z)$, with equality if and only if Z and U are one-to-one. Since we want to minimize $H(X|U)$, we can without loss of generality assume that $U = Z$. From $H(X) = \log m$

we see that X is uniform. It is easy to verify that the only way to achieve that is by setting $p(x|u)$ to be uniform inside the connected component associated with u . This yields

$$\rho_*(G) \geq H(X|U) = \sum_{i=1}^s \frac{m_i \log m_i}{m} = \log m - H(Q).$$

On the other hand, we have $\rho_*(G) \leq \log m - H(Q)$ by Corollary 2. This completes the proof. \blacksquare

Corollary 4. *Let G be a graph with m vertices. Suppose that $C_0(G) = \log s$ for some positive integer s , and there is a unique way (up to permutations) of writing m as a sum of positive integers $m = m_1 + m_2 + \dots + m_t$ such that $H(\frac{m_1}{m}, \dots, \frac{m_t}{m}) = \log m - \rho_*(G)$. If $t = s$ then G is a disjoint union of cliques of sizes m_1, \dots, m_s .*

Proof: From Theorem 10 and the uniqueness assumption it must be that G has s connected components of sizes m_1, \dots, m_s . If even one connected component is not a clique then $C_0(G) \geq \log(s+1) > \log s$, concluding the proof. \blacksquare

Corollary 5. *Let G and H be two graphs with the same number of vertices. Suppose G is a disjoint union of cliques of distinct prime sizes. Assume $C_0(G) = C_0(H)$ and $\rho_*(G) = \rho_*(H)$. Then $G \cong H$.*

Proof: Write $G = K_{p_1} + K_{p_2} + \dots + K_{p_s}$ where $p_1 < p_2 < \dots < p_s$ are distinct primes. Suppose H has t connected components of sizes $m_1 \leq m_2 \leq \dots \leq m_t$. Since the packing points of G and H coincide, by Theorem 10 we have

$$H\left(\frac{p_1}{m}, \dots, \frac{p_s}{m}\right) = H\left(\frac{m_1}{m}, \dots, \frac{m_t}{m}\right). \quad (32)$$

We now show that this entropy equality implies that $s = t$ and $p_i = m_i$ ($1 \leq i \leq s$), which by Corollary 4 will prove our claim.

From (32) we have

$$\prod_{i=1}^s p_i^{p_i} = \prod_{j=1}^t m_j^{m_j}.$$

Fix any j , and let i be such that $p_i | m_j$. Then clearly $p_i^{p_i} | m_j^{m_j}$. Thus for any j there exists a subset $S_j \subseteq [s]$ such that

$$m_j^{m_j} = \prod_{i \in S_j} p_i^{p_i} \quad (33)$$

Moreover, $\{S_1, \dots, S_t\}$ form a partition of $[s]$. Now, if $|S_j| = 1$ for all j then we are done. Suppose to the contrary there exists j such that $|S_j| > 1$. Then (33) implies that $m_j < \sum_{i \in S_j} p_i$. Thus we have $m = \sum_{j=1}^t m_j < \sum_{i=1}^s p_i = m$, in contradiction. \blacksquare

Theorem 11. *Let G be a graph with m vertices. Then G is the disjoint union of s copies of a complete graph K_n , i.e., $G = sK_n$ for some positive integers s and n , if and only if $\rho^*(G) = \rho_*(G)$.*

Proof: If $G = sK_n$, then we can easily verify that $\rho^*(G) = \rho_*(G) = \log n$. Now suppose that $\rho^*(G) = \rho_*(G)$. Assume that G has s connected components of sizes m_1, m_2, \dots, m_s . From Theorem 10 we have

$$\rho^*(G) = \rho_*(G) = \sum_{i=1}^s \frac{m_i \log m_i}{m}.$$

On the other hand, we know $C_0(G) \geq \log s$. For simplicity write $\rho^* = \rho^*(G)$. By the definition of the free-lunch point, we get $C_{\rho^*} = C_0(G) \geq \log s$. Then

$$\log m = \rho^* + C_{\rho^*}(G) \geq \sum_{i=1}^s \frac{m_i \log m_i}{m} + \log s \geq \log m.$$

The above inequality holds if and only if $m_1 = m_2 = \dots = m_s$. This proves the result. \blacksquare

Theorem 12. Let $G = K_{m_1} + K_{m_2} + \cdots + K_{m_s}$ and $H = K_{n_1} + K_{n_2} + \cdots + K_{n_t}$ be two disjoint union of cliques. Suppose that the functions $C_\rho(G)$ and $C_\rho(H)$ coincide. Then $G \cong H$.

Proof: Since the functions $C_\rho(G)$ and $C_\rho(H)$ coincide, the graphs G and H have the same number of vertices, i.e., $m_1 + \cdots + m_s = n_1 + \cdots + n_t$. If $m_1 = \cdots = m_s$, then we get $G \cong H \cong sK_{m_1}$ by Theorem 11. A similar proof applies for the case $n_1 = \cdots = n_t$.

Now assume that $s \geq 2$ and $m_i, 1 \leq i \leq s$ are not all equal and $n_i, 1 \leq i \leq t$ are not all equal. By Theorem 3 we have

$$\begin{aligned}\rho^* &= \rho^*(G) = \rho^*(H) = \frac{1}{s} \sum_{i=1}^s \log m_i = \frac{1}{t} \sum_{j=1}^t \log n_j, \\ \rho_* &= \rho_*(G) = \rho_*(H) = \frac{1}{m} \sum_{i=1}^s m_i \log m_i = \frac{1}{n} \sum_{j=1}^t n_j \log n_j, \\ C_{\rho^*}(G) &= C_{\rho^*}(H) = \log s = \log t.\end{aligned}$$

Hence $s = t$. Fix $\tilde{\rho} \in [\rho^*, \rho_*]$. By (12), there exist $\tilde{\beta}, \tilde{\gamma} \in [0, 1]$ such that

$$\tilde{\rho} = \left(\sum_{i=1}^s m_i^{\tilde{\beta}} \log m_i \right) / \sum_{i=1}^s m_i^{\tilde{\beta}} = \left(\sum_{i=1}^s n_i^{\tilde{\gamma}} \log n_i \right) / \sum_{i=1}^s n_i^{\tilde{\gamma}}.$$

Then, by Theorem 3 and (11), we have

$$\begin{aligned}-\tilde{\beta} &= C'_{\tilde{\rho}}(G) = C'_{\tilde{\rho}}(H) = -\tilde{\gamma}, \\ \log \left(\sum_{i=1}^s m_i^{\tilde{\beta}} \right) &= C_{\tilde{\rho}}(G) + \tilde{\beta}\tilde{\rho} = C_{\tilde{\rho}}(H) + \tilde{\gamma}\tilde{\rho} = \log \left(\sum_{i=1}^s n_i^{\tilde{\beta}} \right).\end{aligned}$$

Therefore

$$\sum_{i=1}^s m_i^\beta = \sum_{i=1}^s n_i^\beta \quad \text{for } 0 \leq \beta \leq 1. \quad (34)$$

Without loss of generality, assume that $m_1 \leq \cdots \leq m_s$ and $n_1 \leq \cdots \leq n_s$. Note that both sides of (34) are analytic functions of β over the whole complex plane. As they coincide in the interval $[0, 1]$, they must be identical over the whole complex plane. Now letting $\beta \rightarrow \infty$ we get $m_s = \max_i m_i = \max_i n_i = n_s$. Applying this argument recursively, we conclude that $m_i = n_i$ for $1 \leq i \leq s$. ■

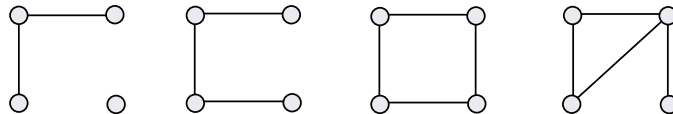
Example 7. We note that the “corner points” of the ρ -capacity curve for a disjoint union of cliques do not always characterize the sizes of the cliques. Let $G = 12K_2 + 6K_8$ and $H = 4K_1 + 13K_4 + K_{16}$. We see that the graphs have the same number of vertices $|V(G)| = |V(H)| = 72$, the same Shannon capacity $C_0(G) = C_0(H) = \log 18$, the same free-lunch point $\rho^*(G) = \rho^*(H) = \frac{5}{3}$, and the same packing point $\rho_*(G) = \rho_*(H) = \frac{7}{3}$, but they are clearly not isomorphic.

VI. OPEN PROBLEMS

Below we mention a few problems of interest.

Problem 1. The ρ -capacity of small graphs.

(i) Find the ρ -capacity of all the graphs with up to 4 vertices. The following four graphs remain unsolved:



(ii) Find the ρ -capacity of the Pentagon C_5 .

Problem 2. Characterize the free-lunch point $\rho^*(G)$. Specifically, give a necessary and sufficient condition for $\rho^*(G) > 0$.

Problem 3. Let G, H be two graphs with $C_\rho(G) = C_\rho(H)$. Do any of the following statements hold?

- 1) If G is a disjoint union of cliques then $G \cong H$.⁴
- 2) If G is a clique minus a clique then $G \cong H$.
- 3) If $E(G) \subseteq E(H)$ then $G \cong H$.
- 4) $G \cong H$.

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APPENDIX

Lemma 7. Let m_1, \dots, m_s be s positive integers, let $m = m_1 + \dots + m_s$, and define a function $g : [-1, 0] \rightarrow \mathbb{R}$ by

$$g(\gamma) = -\log \sum_{i=1}^s m_i^{-\gamma}.$$

Suppose that $s \geq 2$ and m_1, \dots, m_s are not all equal. Then

- 1) The function g is differentiable on $[-1, 0]$ and⁵

$$g'(\gamma) = \left(\sum_{i=1}^s m_i^{-\gamma} \log m_i \right) / \sum_{i=1}^s m_i^{-\gamma},$$

$$g''(\gamma) = -(\log e) \cdot \left(\sum_{1 \leq i < j \leq s} (m_i m_j)^{-\gamma} (\ln m_i - \ln m_j)^2 \right) / \left(\sum_{i=1}^s m_i^{-\gamma} \right)^2.$$

- 2) The function g' is continuous and strictly monotonically decreasing on $[-1, 0]$, and its image

$$g'([-1, 0]) = \left[\frac{1}{s} \sum_{i=1}^s \log m_i, \frac{1}{m} \sum_{i=1}^s m_i \log m_i \right].$$

Proof: The result 1) follows from direct computation. Since $g''(\gamma) < 0$ for $-1 \leq \gamma \leq 0$, we can verify 2) directly. ■

Lemma 8. Let m_1, \dots, m_s be s positive integers, let ρ be a nonnegative number satisfying that

$$\frac{1}{s} \sum_{i=1}^s \log m_i \leq \rho \leq \frac{1}{m} \sum_{i=1}^s m_i \log m_i, \quad (35)$$

and let

$$A(n) = \frac{1}{2^{\rho n}} \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} \leq 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s} m_1^{i_1} \dots m_s^{i_s},$$

$$B(n) = \sum_{\substack{i_1 + \dots + i_s = n \\ m_1^{i_1} \dots m_s^{i_s} \geq 2^{\rho n}}} \binom{n}{i_1, i_2, \dots, i_s}.$$

Suppose that $s \geq 2$ and m_1, \dots, m_s are not all equal. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log A(n) = \lim_{n \rightarrow \infty} \frac{1}{n} \log B(n) = \log \left(\sum_{i=1}^s m_i^\beta \right) - \beta \rho$$

⁴Theorem 12 and Corollaries 4 and 5 address special cases of this problem.

⁵Here e is Euler's number, not the number of edges.

where $\beta \in [0, 1]$ is the unique solution satisfying

$$\rho = \left(\sum_{i=1}^s m_i^\beta \log m_i \right) / \sum_{i=1}^s m_i^\beta.$$

Proof: First, the number of tuples (i_1, \dots, i_s) such that $i_1 + \dots + i_s = n$ is at most $(n+1)^s$. For each tuple (i_1, \dots, i_s) , let $P = (p_1, \dots, p_s) = (i_1/n, \dots, i_s/n)$. Then we have (see [11, Theorem 11.1.3])

$$\frac{1}{(n+1)^s} 2^{nH(P)} \leq \binom{n}{i_1, i_2, \dots, i_s} \leq 2^{nH(P)}.$$

Hence we can reduce the computation of $\lim_{n \rightarrow \infty} \frac{1}{n} \log A(n)$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \log B(n)$ to the following two maximization problems:

$$\begin{aligned} & \text{maximize} && H(P) + \sum_{i=1}^s p_i \log m_i - \rho \\ & \text{subject to} && \sum_{i=1}^s p_i \log m_i \leq \rho \\ & && \sum_{i=1}^s p_i = 1 \\ & && p_i \geq 0, \quad i = 1, \dots, s \end{aligned} \tag{36}$$

and

$$\begin{aligned} & \text{maximize} && H(P) \\ & \text{subject to} && \sum_{i=1}^s p_i \log m_i \geq \rho \\ & && \sum_{i=1}^s p_i = 1 \\ & && p_i \geq 0, \quad i = 1, \dots, s. \end{aligned} \tag{37}$$

Now we will solve the maximization problem (36). We first define the Lagrangian

$$\begin{aligned} & L(p_1, \dots, p_s, \mu_0, \dots, \mu_s, \lambda) \\ &= H(P) + \sum_{i=1}^s p_i \log m_i - \rho - \mu_0 \left(\sum_{i=1}^s p_i \log m_i - \rho \right) - \sum_{i=1}^s \mu_i (-p_i) - \lambda \left(\sum_{i=1}^s p_i - 1 \right) \\ &= H(P) + (1 - \mu_0) \left(\sum_{i=1}^s p_i \log m_i - \rho \right) + \sum_{i=1}^s \mu_i p_i - \lambda \left(\sum_{i=1}^s p_i - 1 \right). \end{aligned}$$

By 2) of Lemma 7 and (35), there exists a unique $\tilde{\mu}_0 \in [0, 1]$ such that

$$\rho = \left(\sum_{i=1}^s m_i^{1-\tilde{\mu}_0} \log m_i \right) / \sum_{i=1}^s m_i^{1-\tilde{\mu}_0}.$$

Let $\tilde{\mu}_i = 0$ for $1 \leq i \leq s$, and let

$$\tilde{p}_i = \frac{m_i^{1-\tilde{\mu}_0}}{\sum_{i=1}^s m_i^{1-\tilde{\mu}_0}}, \quad i = 1, \dots, s,$$

and $\tilde{\lambda} = \log(\sum_{i=1}^s m_i^{1-\tilde{\mu}_0}) - \log e$. Then we can verify that $\tilde{p}_1, \dots, \tilde{p}_s, \tilde{\mu}_0, \dots, \tilde{\mu}_s, \tilde{\lambda}$ satisfy the Karush–Kuhn–Tucker conditions (see [13, Section 5.5.3]). Therefore they are optimal and the maximum is

$$H(\tilde{p}_1, \dots, \tilde{p}_s) = \log \left(\sum_{i=1}^s m_i^{1-\tilde{\mu}_0} \right) - (1 - \tilde{\mu}_0) \rho.$$

Similarly, we can show that these $\tilde{p}_i, i = 1, \dots, s$ are also optimal solutions for the maximization problem (37). Now replacing $1 - \tilde{\mu}_0$ with β will give the result. ■

In the following, we provide the proofs of Lemmas 4-6 of Section IV.

Lemma 4. *Let G be a graph with m vertices and $2 \leq k \leq m$. Suppose $\mathcal{F} = \{V_i \mid 1 \leq i \leq N\}$ is an independent family of G such that $|V_i| \leq k$ for $1 \leq i \leq N$. Then*

$$\sum_{i=1}^N |V_i| \leq \min\{m, (k-1)(2\alpha_k(G) + 1)\}.$$

Proof: The inequality $\sum_{i=1}^N |V_i| \leq m$ is obvious. Now, without loss of generality, we can assume that $|V_i| = k$ for $1 \leq i \leq N_1$ and $|V_i| < k$ for $N_1 < i \leq N$. Then we can obtain a k -independent family from \mathcal{F} as follows. First, set $U_i = V_i$ for $1 \leq i \leq N_1$. Then we define $U_{N_1+1} = \cup_{i=N_1+1}^{N_1+l} V_i$, where l is the smallest integer such that $\sum_{i=N_1+1}^{N_1+l} |V_i| \geq k$. Hence $k \leq |U_{N_1+1}| \leq 2(k-1)$. We continue in this way until the number of vertices contained in the remaining V_i is less than k . Suppose the number of sets U_i we get is equal to M . As these U_i form a k -independent family, we have $M \leq \alpha_k(G)$. It follows that

$$\sum_{i=1}^N |V_i| \leq 2(k-1)\alpha_k(G) + k - 1 = (k-1)(2\alpha_k(G) + 1).$$

Lemma 5. *Let $G = H_1 + H_2 + \dots + H_n$ be the disjoint union of n graphs H_1, \dots, H_n and $k \geq 2$. Then*

$$\sum_{i=1}^n \alpha_k(H_i) \leq \alpha_k(G) \leq \min \left\{ \frac{|V(G)|}{k}, \frac{k-1}{k} \sum_{i=1}^n (2\alpha_k(H_i) + 1) \right\}.$$

Proof: The first inequality can be easily verified, so we only deal with the second. Suppose $\mathcal{F} = \{V_j \mid 1 \leq j \leq N\}$ is an k -independent family of G . Without loss of generality, we can assume that $|V_j| = k$ for all j . Now fix any i . Then $\{V_j \cap V(H_i) \mid 1 \leq j \leq N\}$ is an independent family of H_i . By Lemma 4 we get

$$kN = \sum_{j=1}^N |V_j| = \sum_{i=1}^n \sum_{j=1}^N |V_j \cap V(H_i)| \leq \min \left\{ |V(G)|, (k-1) \sum_{i=1}^n (2\alpha_k(H_i) + 1) \right\}.$$

Now the result follows. ■

Lemma 6. *Let G be a graph. Then for any positive integer $k \leq |V(G)|$, we have $\alpha_k(G) = \alpha_{km}(G \boxtimes K_m)$.*

Proof: Let $\mathcal{F} = \{V_i \mid 1 \leq i \leq N\}$ be a km -independent family of $G \boxtimes K_m$. For each V_i , if a vertex $(u, v) \in V_i$ where $u \in G, v \in K_m$, then without loss of generality we can assume that the set $u \times V(K_m)$ is contained in V_i . Under this assumption, it is not hard to see that there is a one-to-one correspondence between the k -independent family in G and the km -independent family in $G \boxtimes K_m$. The result follows easily from this observation. ■

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